Imperial College London

Lecture 3

Frequency Domain view of signals

Peter Y K Cheung

Dyson School of Design Engineering
Imperial College London

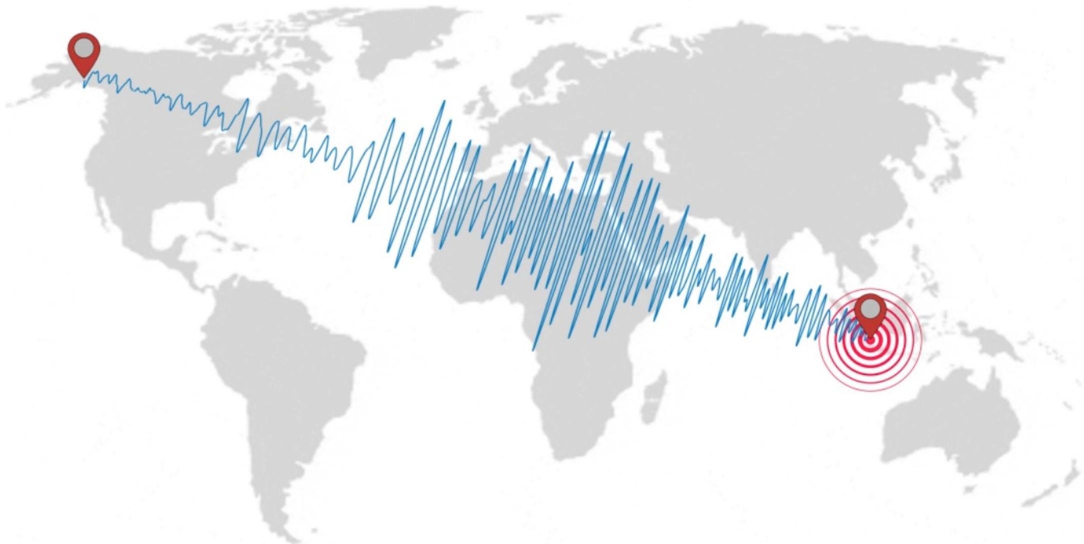
URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

In this lecture, I will be considering signals from the frequency perspective. This is a ***complementary*** view of signals, which consider signals with frequency as the independent variable instead of the time, and is fundamental to the subject of signal processing.

Central to this are two **linear transformations**: the **Fourier Transform** and the **Laplace Transform**. Both will be considered in later lectures.

Remember, linear transforms obey the principle of superposition (as in linear circuits you learned from Electronics 1 last year).

Example – Seimic signal measured in Alaska due to Sumatra Earthquake in 2004



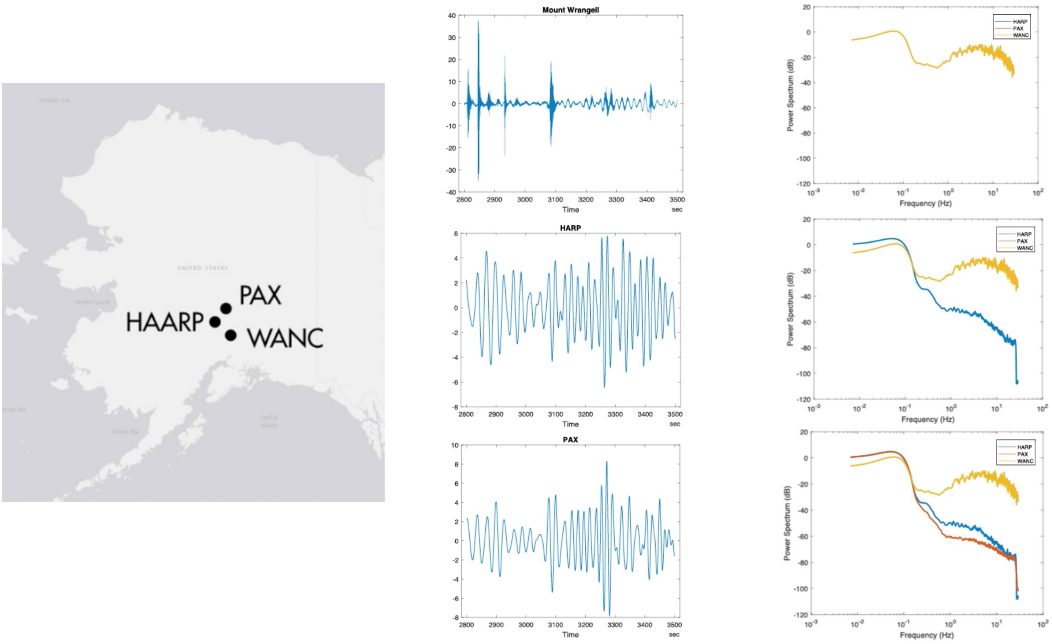
PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 2

Let us take a concrete example. 2004 earthquake in Sumatra in Indonesia caused seismic wave travelled around the world and was recorded at three geostations in Alaska. These were recorded as electricals signals and available for scientist to analyse and understand the impact of earthquakes.

This example is taken from MathWork's excellent online tutorial that you can take if you wish. The link is given below, but you would need to be a registered user of Matlab before you can gain access.

https://matlabacademy.mathworks.com/details/signal-processing-onramp/signalprocessing

Three stations recorded data for analysis

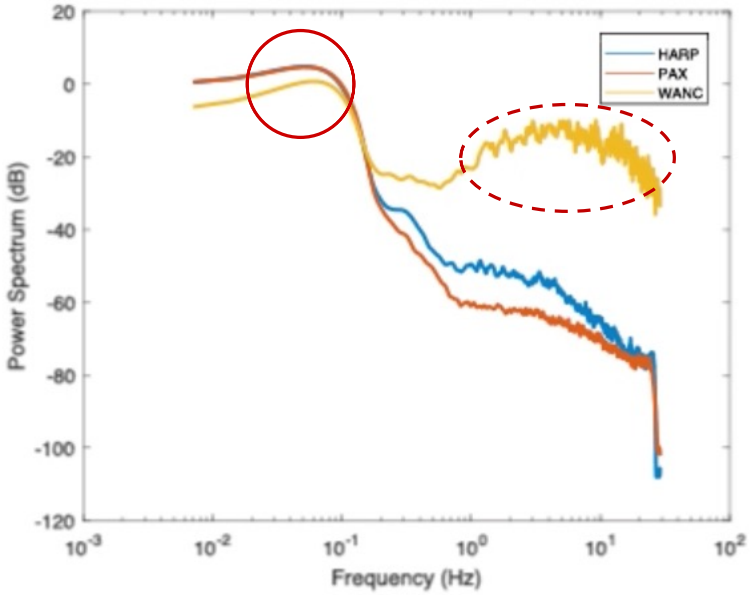


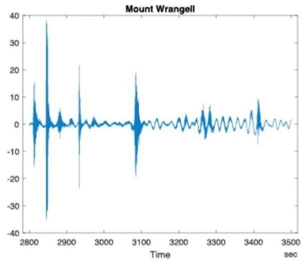
The earthquake data at the three states looks very different as shown in the slides.

The time domain signal recorded at the WANC station looked very different from those from PAX and HAARP stations. It is a great example to demonstrate the limitation of time domain view of a signal.

However, if we plot the power spectrum of these signals, the truth starts to emerge!

Frequency domain view of signal – more informative

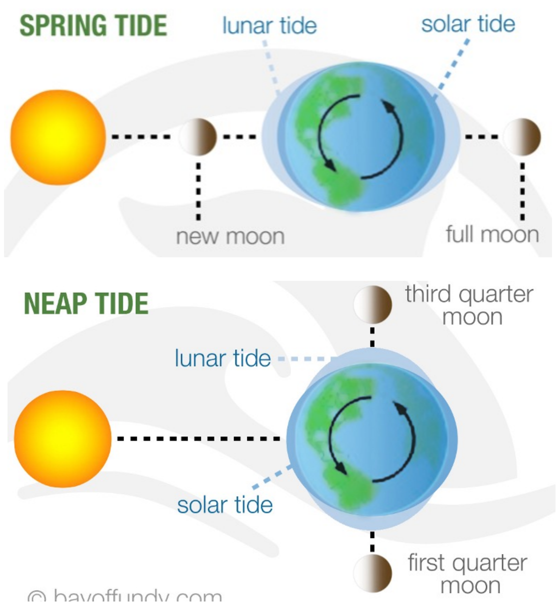




All three stations in fact captured similar low frequency activities (<0.1Hz) while only the WANC station managed to capture some high frequency activities (at around 10HZ) as well. Further, this high frequency activies swarmed the time domain signal, which made it rather difficult to interprete what's going on.

This example illustrates the value of analysing signals in frequency instead of time domain.

Prediction of Tides



- Early scientists:
  - Isaac Newton
  - Joseph Fourier
  - William Thomson (Lord Kelvin)
  - Pierre-Simon Laplace

http://oceanmotion.org/html/background/tides-observing.htm

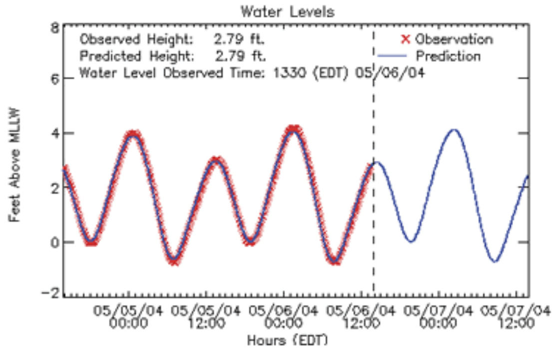
Frequency analysis (also known as harmonic analysis) was explored centuries ago, initially motivated by the understanding and prediction of the tides. The foundational work was done by Isaac Newton, whose gravitation formulation explained why the Moon would exert attractive forces on the ocean to cause the water level to go up and down depending on Moon's distance from Earth. Since Earth rotates once per 24 hours, and Moon rotates around Earth once in 28 days, the periodicity of the tides due to the Moon (lunar tides) is 24 hours 50 minutes. (The extra 50 minutes is because by the time Earth rotated once, Moon has moved 1/28 of its orbit.)

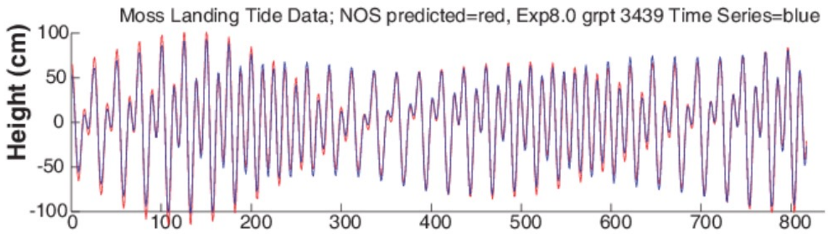
The Moon is not the only cause of the tides. The Sun, though much further away, is much larger than the Moon. It also exerts a gravitational force on the ocean.

There are many many other factors that affect the level of water in the ocean at a particular location, but Moon, Sun and Earth's rotational orbits, which are ALL periodical, contribute most to the tides.

Fourier's harmonic analysis was used by **William Thomson** to perform tide prediction while **Pierre Laplace** derived the equations to model the dynamic behaviour of the flow of water due to these forces.

Periodicity of the Tides



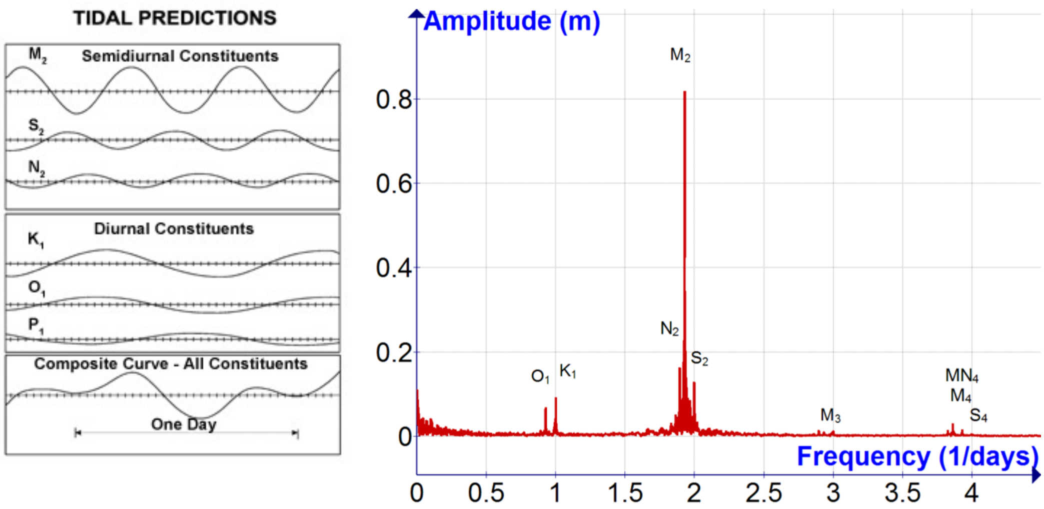


http://oceanmotion.org/html/background/tides-observing.htm

A plot of the water level vs time shows the periodicity clearly. There are two high tides in roughly a 24 hours (+ 50 minutes) period. The peak levels varies depending on whether it is the first tide or the second tide of the day. The higher level corresponds to the Moon being directly above the location on Earth, while the second high tide corresponds to the Moon being on the opposition side (and the swell is due to centrifugal force caused by the spin of Earth and the Moon).

Zooming out in time (lower plot where x-axis is in hours) shows the effect of the Sun on the tides during a typical month.

Tides decomposed into periodic constituents



Therefore each factor that affects the tides at a particular location on Earth can be model as a sinewave with a particular frequency and phase (relative to others). The amplitude of the sinewave is a measure of how strong that factor is on the tide. If we have a good model for each factor, tides can be predicted by adding all these sinewave together. This is the basis of **Fourier** or **Harmonic analysis**.

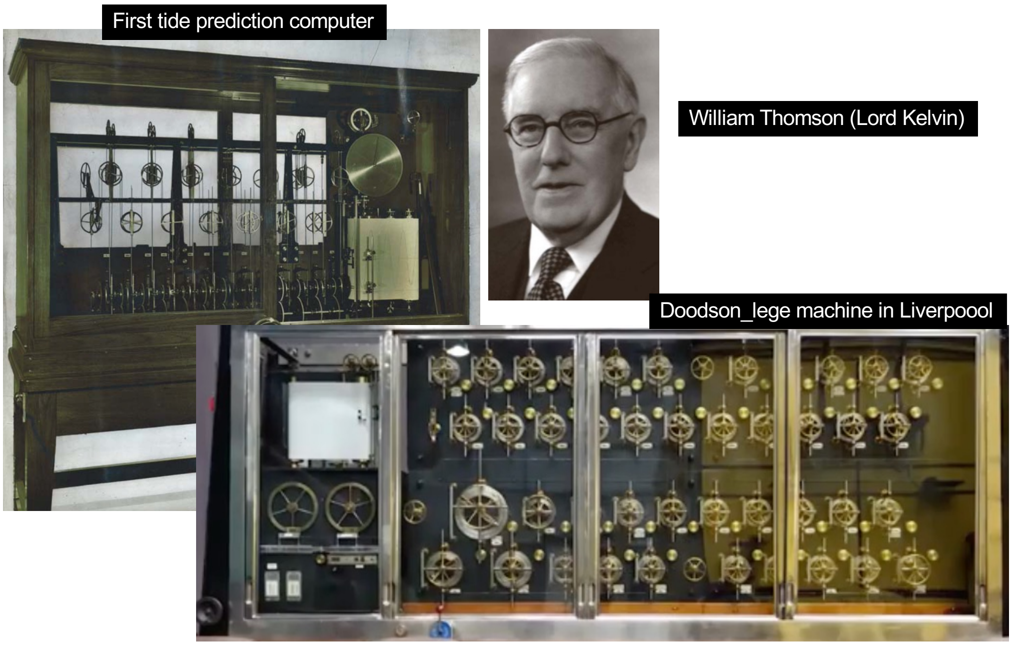
As shown above, there are many such factors but five or six of these dominate. If we take real-life tides data and calculate the frequency spectrum of the water level, we obtain the amplitude (water level) vs frequency. There are many peaks. The most significant is M2 – this is the effect of the Moon alone. Its periodicity is just below 2 per day (24 hours 50 minutes). The second most significant (for this location) is N2. This is due to the fact that the orbit of the Moon around the Earth is not a perfect circle, but an ellipse. Further, the Moon does not rotate around the Earth as its centre. That deviation resulted in N2. Finally S2 accounts for the bulk of the effect of the Sun on the tides and has a period of 2 per day.

The Moon also affects the tide in a way that it repeats each time the Earth spins, causing O1 and K1. There are many other constituent components, but they are much small.

According to NOAA of the US Government, there are 37 constituent components that can be take into account.

https://tidesandcurrents.noaa.gov/harcon.html?id=9410170

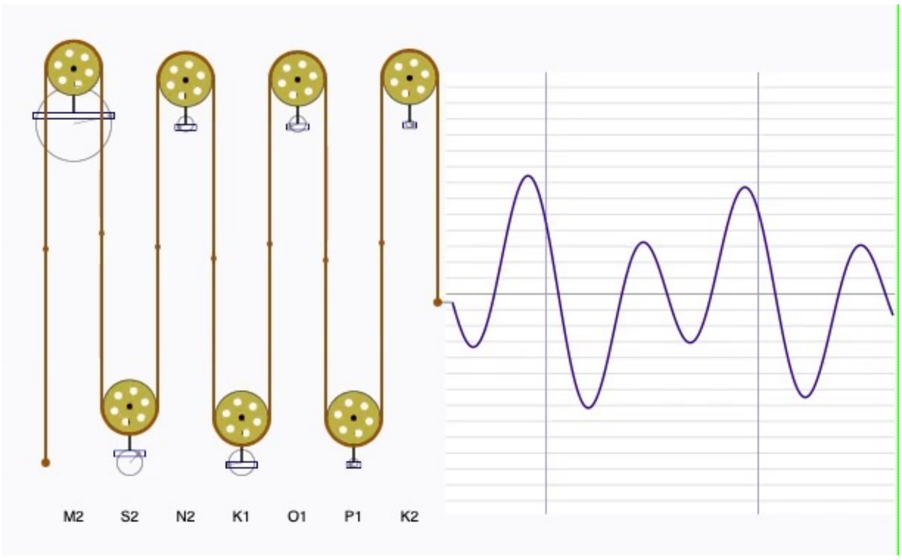
Kelvin's Tide Prediction Machine



Lord Kelvin (William Thomson) was the first to build a mechanical machine (or analogue computer) to perform prediction of the tides. Each pulley model one harmonic factor. The summation is performed by the strings on the pulley which is attached to a pen on the right end. This pen draws the predicted level on a drum of paper which rotates. In this way, a plot of tide level vs time can be obtained.

Years later, A.T. Doodson refined the design to produce a much better prediction machine which included many more constituent components. This tide prediction machine can be seen working in the Science Museum in Liverpool.

Animation of the Tide Prediction Machine



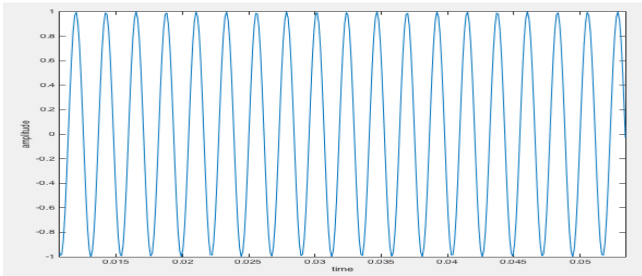
https://www.ams.org/publicoutreach/feature-column/fcarc-tidesiii3

Here is a animation of the Tide Prediction Machines with 7 constituent components creating a prediction. Time travels from right to left.

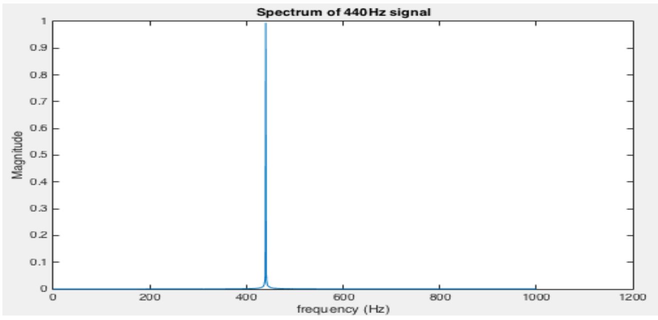
You can try out this animation written in Java here:
https://www.ams.org/publicoutreach/feature-column/fcarc-tidesiii3

Time vs Frequency view of a sinewave

- Sinewave (sinusoidal signal) in time domain



- Same sinewave in frequency domain



If we have a time domain sinewave as shown here, storing this signal takes a lot of memory. You would need to sample the signal frequency at different time points.

However, if you consider this as a plot of amplitude vs frequency, it appears as an impulse function at a certain frequency value – one single amplitude! Isn't this more simple and clear?

Actually, not shown here is the phase information of the sinewave, which is determined by the starting phase of the sinusoidal function at t=0. This is the frequency view of the sinewave signal.

In other words, you can describe a sinewave by its frequency, its phase and its amplitude:

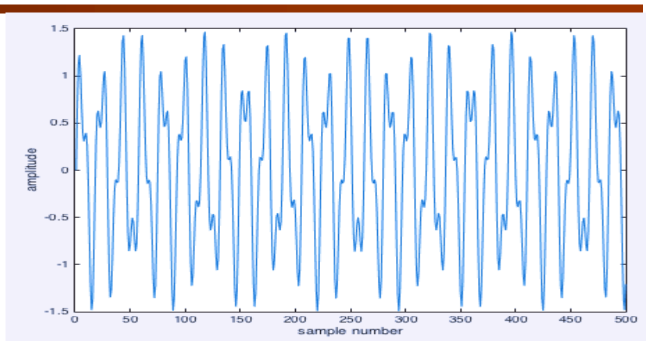
$$x(t) = \boldsymbol{A}\sin(\boldsymbol{\omega} t + \boldsymbol{\phi})$$

Since cos x = sin (x + $\frac{\pi}{2}$) , we can equally characterize a sinusoidal signal as a cosine function:

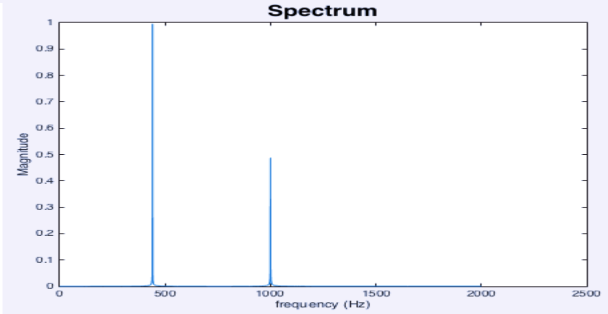
$$x(t) = \boldsymbol{A}\cos(\boldsymbol{\omega} t + \boldsymbol{\phi} - \frac{\pi}{2})$$

Two sinewaves

- Adding 440Hz to 1kHz signal. The 440Hz is twice as large as the 1kHz signal.



- Spectrum of two sinewaves

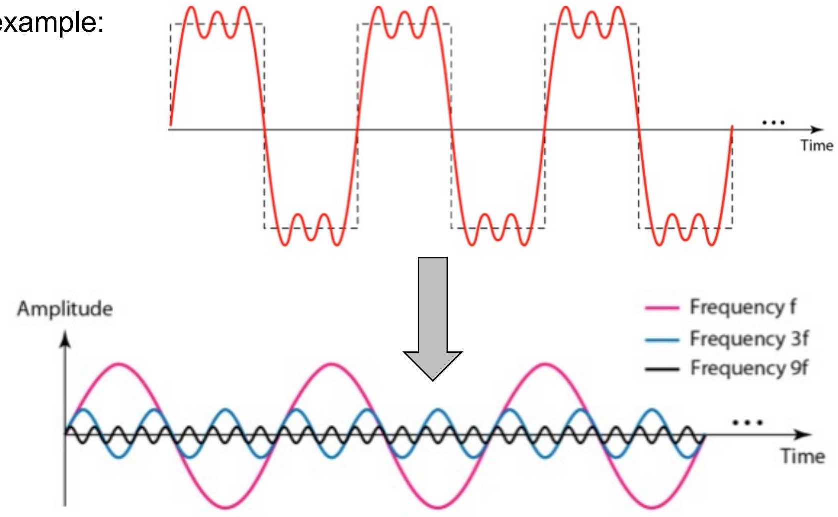


Now if we add a sinewaves at 440Hz and to another at 1000Hz (at half amplitude of the 440Hz), we get a more complex time domain waveform as shown.

It is not obvious what made up this time domain signal. However, view in the frequency domain, we clearly see two impulses at the corresponding frequencies, one half the height of the other.

Key idea – Fourier's theory

- Basic idea – **any time domain signal can be constructed from weighted linear sum of sinusoidal signals** (sine or cosine signals) at different frequencies.
- For example:



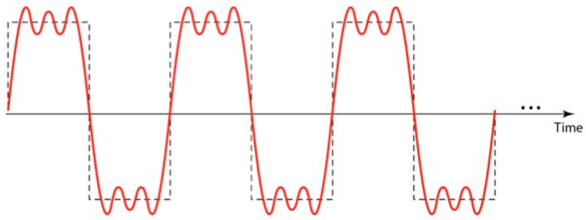
In fact, Fourier has long proven that any time domain signal can be represented as a linear combination of sinewaves at different frequencies, suitably weighted by some coefficients. Shown here are three sinewaves at f, 3f and 9f, of different amplitudes. When added together, we get the approximation of a square wave shown in RED.

When a time-domain signal is resolved into various frequency components, we call the plot of amplitude vs frequency the "**amplitude spectrum**" of the signal.

Spectrum – Frequency domain representation

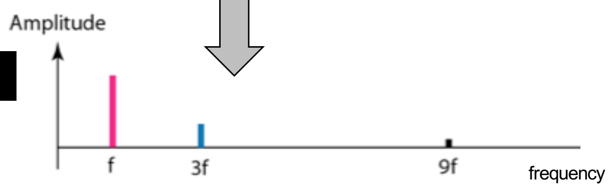
- Instead of having to store individual time samples, we only need to store the amplitude, frequency and phase of each sinusoidal signal.

Time domain



- Spectrum of signal in frequency domain is represented by amplitude value for each frequency. There is also phase vs frequency, which is not shown here.

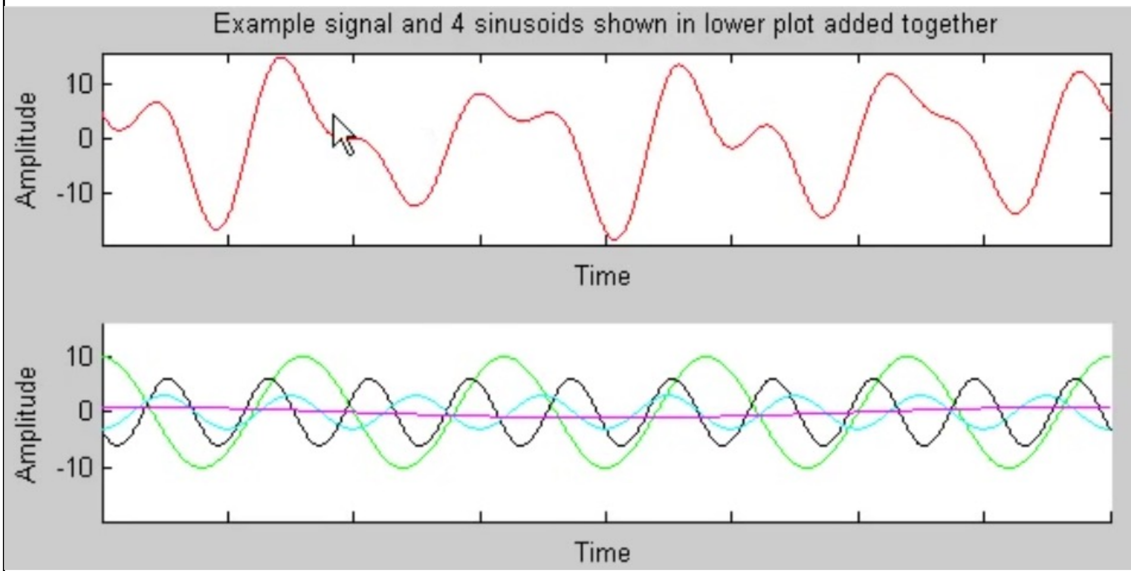
Frequency domain



Now instead of complicated time domain sample of the approximate square wave signal, we only need to use three values at f, 3f and 9f. Here we assume all the phases are zero, i.e. all the three sinusoids start as 0 at t=0.

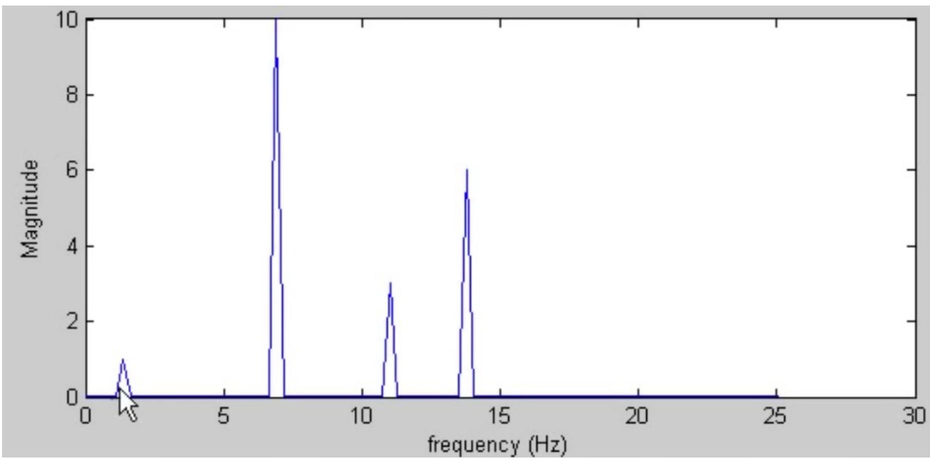
Another Example

- Here is another time domain signal that is constructed with four sine waves:



Here is another example of a signal made out of four sinusoids.

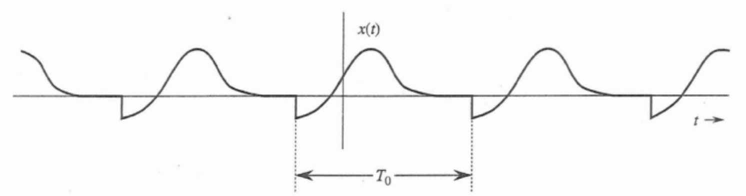
The important message here is that we can do the inverse. Instead of constructing the time domain signal by combining the sinusoidal waveforms, we can go the other way: take a time-domain signal, and divide this (or resolve it) into various sinusoidal signals, at different frequencies with different amplitude and phase. This is the foundation of **Fourier transform**.



Periodic Signal & Fourier Series

- A periodic signal $x(t)$ with a period of To has the property:

  $x(t) = x(t + T_0)$ for all t



- **Fourier series** expresses $x(t)$ as a **weighted linear sum of sinusoids** (or expontentials) of the fundamental frequency $f_0 = 1/T_0$ and all it harmonics nf_0 where n = 2, 3, 4…..

  $$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{for all integers n}$$

- ω_0 is called the **fundamental frequency** such that (f_0 in cycles/sec or Hz, ω_0 in radians/sec)

  $\omega_0 = 2\pi f_0 = 2\pi / T_0$ and $n\omega_0$ are the harmonic frequencies

- a_0 is the DC (mean) value of $x(t)$ and a_n , b_n are the Fourier coefficients at the frequency $n\omega_0$

PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 15

You have been taught Fourier series in first year maths course. Here is a quick revision.

A periodic signal (or function) is one that repeats itself in time. The interval between repetition is the period To.

A periodic signal x(t) can be mathematically written as a sum of sine and cosine waves with different weighting (or coefficients). The frequencies of the sine/cosine waves is determined by the periodicity of the signal To, which determines the fundamental frequency fo = 1/To.

Remember frequency can be expressed as cycles/second (Hz) or as radians/sec.

How to find a_0?

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- To determine a_0, we multiply both sides by $\cos m\omega_0 t$ and intergrate over T_0:

$$\int_0^{T_o} x(t)dt = a_0 \int_0^{T_o} dt$$
$$+\sum_{n=1}^{\infty} a_n \int_0^{T_o} \cos n\omega_0 t \ dt$$
$$+\sum_{n=1}^{\infty} b_n \int_{T_o}^{T_o} \sin n\omega_0 t \ dt$$

- 2nd and 3rd terms integrates to zero over one period of time. Therefore only the first term survives:

$$\int_0^{T_o} x(t)dt = a_0 \int_0^{T_o} dt = a_0 \, T_o$$

- Therefore

$$a_0 = \frac{1}{T_o}\int_0^{T_o} x(t)dt$$

Let us now explore how to compute the various coefficients in the Fourier series.

To do this, we simply integrate both sides of Fourier series equation. Since the right-hand-sides of the equations consists of linear sum of terms, we can perform the integration operation one term at a time.

All the cosine and sine functions must integrate to zero over one entire period of the signal. Therefore the only term remaining is the a_0 term! This is simply the average value (or DC) of the signal, which can be calculated by integration over one period of the signal.

How to find a_n and b_n coefficients? (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\,)$$

- To determine a_0, we simply intergrate both sides of the equation over one period T_0:

$$\int_0^{T_o} x(t) \cos m\omega_0 t\, dt = a_0 \int_0^{T_o} \cos m\omega_0 t\, dt$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_o} \cos n\omega_0 t\, \cos m\omega_0 t\, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{T_o}^{T_o} \sin n\omega_0 t\, \cos m\omega_0 t\, dt$$

- But:

$$\int_0^{T_o} \cos m\omega_0 t\, dt = 0 \quad \text{and} \quad \int_0^{T_o} \cos n\omega_0 t\, \cos m\omega_0 t\, dt = 0 \quad \text{if} \quad n \neq m$$

- When n = m,

$$\int_0^{T_o} \cos m\omega_0 t\, \cos m\omega_0 t\, dt = \frac{T_0}{2}$$

PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 17

The a_n coefficients can be computed by multiplying both sides by the function $\cos m\omega_0 t$ and then perform integration over one period of the signal.

The a_0 term integrates to zero.

The $\cos n\omega_0 t\ \cos m\omega_0 t$ term also integrates to zero because of the trigonometric identify: $\cos x\ \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$ for all n ≠ m.

However for n = m, the integral is:

$$\int_0^{T_o} \cos m\omega_0 t\ \cos m\omega_0 t\, dt = \int_0^{T_o} \frac{1}{2}[1 + \cos 2m\omega_0 t]dt = \frac{T_0}{2}$$

How to find a_n and b_n coefficients? (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\,)$$

- Therefore, the ONLY term that survives after multiply by $\cos m\omega_0 t$ and integration is:

$$\int_0^{T_o} x(t) \cos m\omega_0 t \, dt = a_m \frac{T_0}{2}$$

- Hence, $a_n = \frac{2}{T_0} \int_0^{T_o} x(t) \cos n\omega_0 t \, dt$ (m = n)
- Similarly to find b_n multiply x(t) by $\sin m\omega_0 t$ and integration over T_o:

$$\int_0^{T_o} x(t) \sin m\omega_0 t \, dt = b_m \frac{T_0}{2}$$

- Hence, $b_n = \frac{2}{T_0} \int_0^{T_o} x(t) \sin n\omega_0 t \, dt$

PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 18

Therefore the only surviving term of these operations is when n = m.

The significant implication of this derivation is:

To find the coefficient a_n of the nth harmonic frequency component, mutilply the signal x(t) with $\cos n\omega_0 t$ and integrate over one period. The result after multiplying by 2/To is the coefficient.

Similar, we can find b_n with the same operation, but now we multiple the signal with sine function instead.

Compact form of Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\,)$$

- A more compact form of the Fourier Series is derived with the trigonometric identity:

$$C\cos(\omega_0 t + \theta) = C\cos\theta\cos\omega_0 t - C\sin\theta\sin\omega_0 t$$
$$= a\cos\omega_0 t + b\sin\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\,)$$
$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\,n\omega_0 t + \theta_n)$$

where

$$C_0 = a_0$$ DC term

$$C_n = \sqrt{a_n{}^2 + b_n{}^2}$$ amplitude

$$\theta_n = tan^{-1} - \left(\frac{b_n}{a_n}\right)$$ phase angle

PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 19

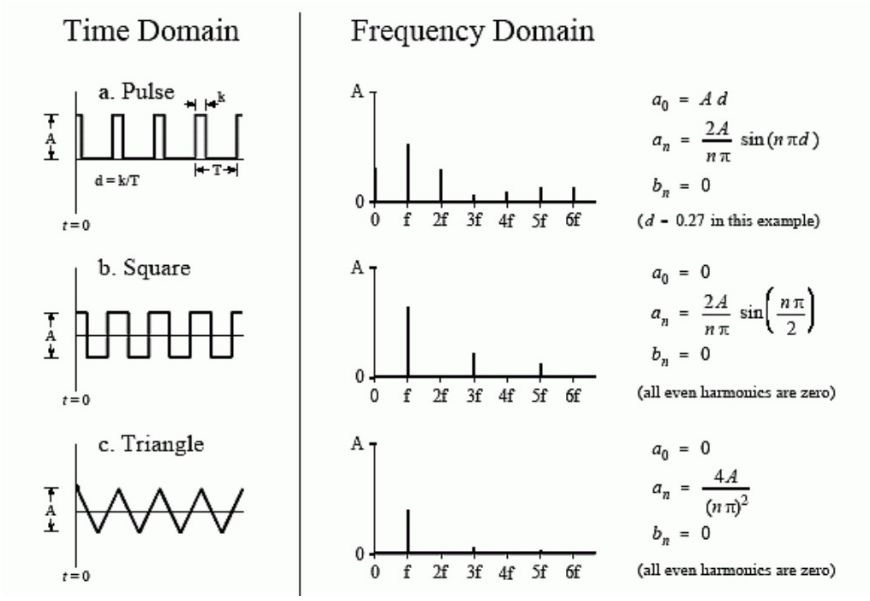
Using the trigonometric identities, we can combine the sine and cosine terms in the Fourier series as shown in the slide above.

Why is this useful? Now we express the Fourier series using one coeffient (C_n) instead of two (a_n and b_n).

C_n is the amplitude of the signal at frequency $n\omega_0$. A plot of C_n at different harmonic frequencies gives us the amplitude spectrum.

θ_n is the phase of the signal at frequency $n\omega_0$. A plot of θ_n at different harmonic frequencies gives us the phase spectrum.

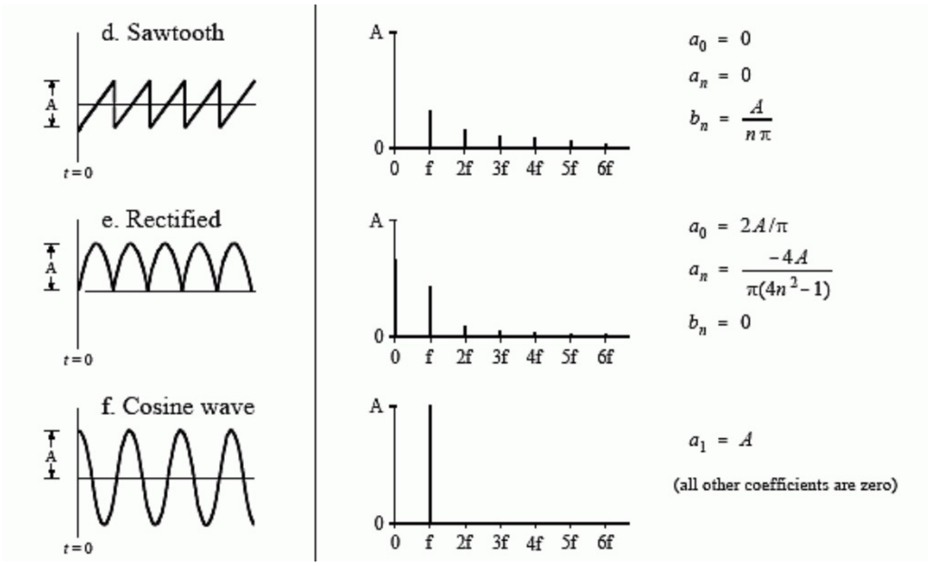
Fourier Series of common signals (1)



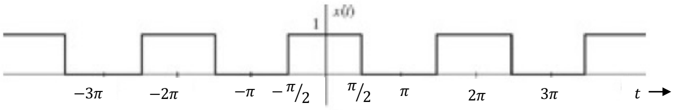
PYKC 15 Jan 2024 DESE50002 - Electronics 2 Lecture 3 Slide 20

These two slides shows the Fourier series coefficients for a number of common signals you might encounter. It is provide here as useful reference.

Fourier Series of common signals (2)



Fourier series of an even signal



- The Fourier series for the square-pulse periodic signal shown above is:

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t\ + \ .....\right)$$

- The symmetry of this even signal result in three properties:

1. Such symmetry implies an even even function. Therefore the Fourier series representation only has cosine terms which are also even functions.
2. This symmetry at t = 0 also result in phase angle at all harmonic frequencies = 0.
3. It only has odd harmonic components – no even harmonic components.

Consider a periodic pulse signal between 0V and 1V with a period of π and is symmetrical about t = 0. The pulse signal has a duty cycle of 50% (or mark-space ratio of 1).

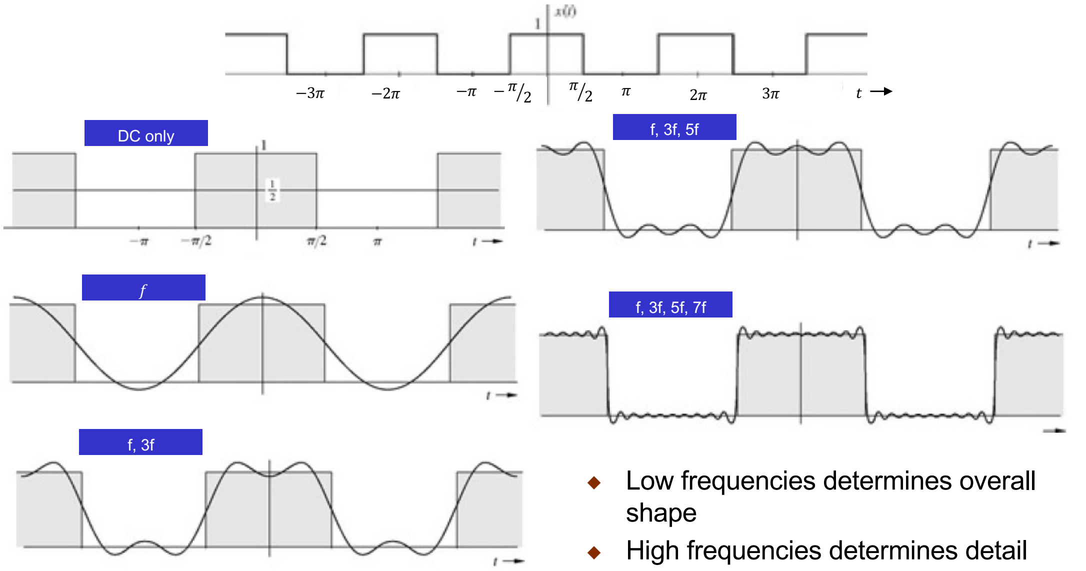
Such a signal has a Fourier series as shown – the derivation of this is left as a tutorial problem. You should be able to apply the formula in the previous slides to derive this equation.

The above three observations are use. The symmetry at t = 0 means that sine components are not present – only cosine terms are left. Why? Inclusion of any sine terms will destroy the symmetry.

All the cosine terms must have a phase of 0. If not, you will also not get the symmetry shown here.

Finally the Fourier series only contains odd harmonics (i.e. 1, 3, 5 etc.) Even harmonics will also destroy the symmetry (or evenness of the signal)!

Fourier coefficiences and waveshaping



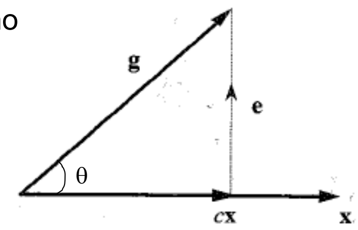
- Low frequencies determines overall shape
- High frequencies determines detail structures

What are the physical significance of the various harmonic component on the shape of the time domain waveform? Here we gradually increase the number of frequency components from DC (0Hz) up, adding one extra harmonic component each time. It is clear that the lower frequencies (DC term and the fundamental frequency component) already give the basic shape of the waveform. Adding more and more harmonic components enhances the details including the edges of the signal.

A Vector view of Signal

- To understand why a signal can be represented by linear sum of sinusoidal waveforms, it is useful to consider electrical signals as VECTORS.
- A vector is specified by its magnitude (or length) and its direction.
- Consider two vectors $\mathbf{g}$ and $\mathbf{x}$. If we project $\mathbf{g}$ onto $\mathbf{x}$, we get c$\mathbf{x}$, where c is a scalar (i.e. constant with no direction).
- If we approximate $\mathbf{g}$ with c$\mathbf{x}$, then

  $\mathbf{g} = c\,\mathbf{x} + \mathbf{e}$
- $\mathbf{e}$, the error vector, is minimum when it is perpendicular to $\mathbf{x}$.
- c$\mathbf{x}$ is known as the **projection** of $\mathbf{g}$ onto $\mathbf{x}$.



- It can be shown (in the notes below) that:

$$\boxed{c = \frac{\mathbf{g}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} = \frac{1}{|\mathbf{x}|^2}\,\mathbf{g}\cdot\mathbf{x}}$$

Dot product

$$\boxed{\mathbf{g}\cdot\mathbf{x} = |\mathbf{g}||\mathbf{x}|\cos\theta}$$

Another useful view of signals is to regard them as **vectors**.

If we have a signal (vector) $\mathbf{g}$, we can ask the question: how much $\mathbf{g}$ is in common with the vector $\mathbf{x}$? The answer is to find the **projection** of $\mathbf{g}$ onto $\mathbf{x}$.

If we want to approximate the vector $\mathbf{g}$ with $\mathbf{x}$, we can write: $\mathbf{g} = c\,\mathbf{x} + \mathbf{e}$ where c is a constant value, and $\mathbf{e}$ is the error, which is minimized when $\mathbf{e}$ is perpendicular to $\mathbf{x}$.

This idea of **projection** is important. If $\mathbf{g}$ contains NO component similar to $\mathbf{x}$, then $\mathbf{g}$ is **perpendicular** to $\mathbf{x}$, and the projection is 0 and $c = 0$.

The projection of $\mathbf{g}$ onto $\mathbf{x}$ is also mathematically express as the "**dot product**"

$$\mathbf{g} \bullet \mathbf{x} = |\mathbf{g}|\,|\mathbf{x}|\ \cos\theta$$

If two vectors have zero dot-product (or the projection of one onto the other is zero), they are known as **orthogonal** to each other.

To calculate an express for the constant c, simple trigonometric gives:

$$c\,|\mathbf{x}| = |\mathbf{g}|\cos\theta$$

Therefore

$$c\,|\mathbf{x}|^2 = |\mathbf{g}|\,|\mathbf{x}|\cos\theta = \mathbf{g} \bullet \mathbf{x}$$

Hence,

$$c = \mathbf{g} \bullet \mathbf{x} \,/\, \mathbf{x} \bullet \mathbf{x} = \frac{1}{|x|^2}\ \mathbf{g} \bullet \mathbf{x}$$

Orthogonal Set of signals

- If vector **g** is at right angle to vector **x**, then the projection of **g** and **x** is zero. These two vectors (or signals) are known to be **orthogonal**.
- It can easily be shown that two sinusoidal signals of DIFFERENT frequencies are orthogonal to each other.
- The complete set of sinusoidal signals (i.e. of all possible frequency) forms a COMPLETE orthogonal set of signals.
- What this means is that ALL time domain signals can be formed out of projects (or components) onto these these sinusoidal set of signals!
- This is the foundation of **Fourier Series** and **Fourier Transform**, which will be discussed further at the next Lecture.

This idea of projection is important. Because as it turns out, the sinusoidal signals (of all frequencies) has two interesting characteristics:

1. Two sinusoidal signals of different frequencies are orthogonal to one another.
2. The set of all sinusoidal signals form a complete or closed set. This means that you can always resolve any time domain signals into sum of projects to the set of all sinusoidal signals.

This is why sinusoidal signals are so important, and why Fourier transform exists!

Three Big Ideas

1. **Time domain** view of a signal is often insufficient. It is often more informative to consider how the signal would appear as a function of frequency, in the **frequency domain**.
2. Any time varying signal can be **decomposed into sinusoidal constituent components** of specific frequencies, phases, and amplitudes, just like the tidal level. This is the main idea of Fourier.
3. Two sinusoidal signals of different frequencies are **orthogonal** to each other, meaning that they have nothing in common, and it is not possible to "produce" one from the other through any linear methods.

This idea of projection is important. Because as it turns out, the sinusoidal signals (of all frequencies) has two interesting characteristics:

1. Two sinusoidal signals of different frequencies are orthogonal to one another.
2. The set of all sinusoidal signals form a complete or closed set. This means that you can always resolve any time domain signals into sum of projects to the set of all sinusoidal signals.

This is why sinusoidal signals are so important, and why Fourier transform exists!